M337/Specimen exam 2

Module Examination

Complex analysis

There are **two parts** to this examination.

In **Part 1** you should **submit answers to <u>all</u> 6 questions**. Each question is worth 10% of the total mark.

In **Part 2** you should **submit answers to <u>2 out of the 3</u> questions**. Each question is worth 20% of the total mark.

Do not submit more than two answers for Part 2. If you submit answers to all three Part 2 questions, then only Questions 7 and 8 will be marked.

Include all your working, as some marks are awarded for this.

Write your answers in **pen**, though you may draw diagrams in pencil.

Start your answer to each question on a new page, clearly indicating the number of the question.

Crossed out work will not be marked.

Follow the instructions in the online timed examination for how to submit your work. Further information about completing and submitting your examination work is in the *Instructions and guidance for your remote examination* document on the module website.

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Part 1

You should **submit answers to all questions** from Part 1.

Each question is worth 10%.

Question 1

Express each of the following complex numbers in *polar* form, simplifying your answers as far as possible.

(a) $1 - i$ [2]

(b) $(1 - i)^6$ [2]

(c) i^{1-i} [3]

(d) $\dfrac{\sqrt{3} + i}{1 - i}$ [3]

Question 2

(a) Evaluate the following integrals with $C = \{z : |z| = 1\}$. Name any standard results that you use and check that their hypotheses are satisfied.

(i) $\displaystyle\int_C \frac{\cos z}{z - 3}\,dz$ [2]

(ii) $\displaystyle\int_C \frac{\cos z}{z(z - 3)}\,dz$ [2]

(iii) $\displaystyle\int_C \frac{\cos z}{z^3(z - 3)}\,dz$ [4]

(b) Explain how the three answers you obtained in part (a) would change if C was replaced by the contour $\Gamma : \gamma(t) = e^{it}$ $(t \in [0, 4\pi])$. [2]

Question 3

(a) Find the residue of the function

$$f(z) = \frac{z^2 + 2}{z^4 - 1}$$

at each of its poles. [4]

(b) Use part (a) to evaluate the integral

$$\int_\Gamma \frac{z^2 + 2}{z^4 - 1}\,dz,$$

where Γ is the circle $\{z : |z| = 2\}$. [2]

(c) Use part (a) to evaluate the real improper integral

$$\int_{-\infty}^{\infty} \frac{t^2 + 2}{t^4 - 1}\,dt.$$ [4]

Question 4

Let f be the Möbius transformation

$$f(z) = \frac{2}{z - i},$$

and let $C = \{z : |z + 1| = 1\}$.

(a) Find the point β such that $\alpha = \infty$ and β are inverse points with respect to C. [2]

(b) Determine an equation for the image circle $f(C)$ in Apollonian form. [5]

(c) Find the centre and radius of $f(C)$. [3]

Question 5

Let q be the velocity function

$$q(z) = \frac{i}{\overline{z}^2}.$$

(a) Determine the largest region in the complex plane on which q represents an ideal flow. [2]

(b) Find a stream function Ψ for this flow, and use the equations

$$\Psi(x + iy) = k, \quad \text{for real constants } k,$$

to find equations for the streamlines for the flow. [4]

(c) Sketch some streamlines to show the nature of the flow, and indicate the direction of flow on each streamline. [4]

Question 6

(a) Let f be the function

$$f(z) = z^2 - z.$$

(i) Find the fixed points of f and classify them as attracting, repelling or indifferent, identifying any attracting fixed points that are super-attracting. [5]

(ii) Find the solutions of the equation $f^2(z) = z$ and hence prove that f has no periodic points of period 2. [3]

(b) Determine whether or not the point $-\sqrt{3}$ lies in the Mandelbrot set. [2]

Part 2

You should **submit answers to two questions** from Part 2. If you submit answers to all three Part 2 questions, then only Questions 7 and 8 will be marked.

Each question is worth 20%.

Question 7

(a) (i) Use the Cauchy–Riemann Theorem to show that there is no point $z = x + iy$ of $\mathbb{C}$ at which the function

$$f(x + iy) = e^y(\cos x + i \sin x)$$

is differentiable. *[7]*

(ii) Determine all the points $z = x + iy$ of $\mathbb{C}$ at which the function

$$g(x + iy) = e^x(\cos y + i \sin y)$$

is differentiable. *[2]*

(b) Let Γ_1 and Γ_2 be the smooth paths

$$\Gamma_1 : \gamma_1(t) = e^{it} \quad (t \in [0, 2\pi]),$$
$$\Gamma_2 : \gamma_2(t) = (t - 1) + it \quad (t \in \mathbb{R}).$$

(i) Show that Γ_1 and Γ_2 meet at the point i, and find the angle from Γ_1 to Γ_2 at i. *[4]*

(ii) Sketch Γ_1 and Γ_2 on the same diagram, indicating the directions of increasing values of t. *[3]*

(iii) Let g be the function

$$g(z) = \exp(z^2).$$

Determine all the points of the complex plane at which g is conformal. *[2]*

(iv) Find the angle from the path $g(\Gamma_1)$ to the path $g(\Gamma_2)$ at the point $g(i)$. *[2]*

Question 8

(a) Find the Taylor series about 0 for the function

$$f(z) = \exp((1 + z)^{-1/3} - 1),$$

up to the term in z^3. *[7]*

(b) Find the Laurent series about -1 for the function

$$g(z) = \frac{5}{(z - 3)(z + 2)}$$

on the annulus $\{z : 1 < |z + 1| < 4\}$, giving the constant term and two terms on each side of it. *[10]*

(c) Let f_1 and f_2 be entire functions such that $f_1(x) = f_2(x)$ for each rational number x. Prove that f_1 and f_2 are equal. *[3]*

Question 9

(a) (i) Use the Taylor series about 0 for cosh to prove that

$$|\cosh z - 1| < 1, \quad \text{for } |z| = 1.$$ *[4]*

(ii) Use your answer to part (a)(i) and Rouché's Theorem to find all the solutions of the equation

$$\cosh z = 1 + z$$

in the open unit disc $\{z : |z| < 1\}$. *[6]*

(b) Determine

$$\max\{|z^2 \exp(1 + z^2)| : |z| \leq 2\},$$

and find all points at which the maximum is attained, giving your answers in Cartesian form. *[10]*

[END OF QUESTION PAPER]